Note Title 13/07/2010

# C3 Exercise 6C (reciprocal trig. identities & equations)

1 Rewrite the following as powers of $\sec\theta$, $\operatorname{cosec}\theta$ or $\cot\theta$.

a $\dfrac{1}{\sin^3\theta} \equiv \left(\dfrac{1}{\sin\theta}\right)^3 \equiv \operatorname{cosec}^3\theta$ ✓

b $\sqrt{\dfrac{4}{\tan^6\theta}} \equiv \dfrac{\sqrt{4}}{\sqrt{\tan^6\theta}} \equiv \dfrac{2}{\tan^3\theta} \equiv 2\left(\dfrac{1}{\tan\theta}\right)^3 \equiv 2\cot^3\theta$ ✓

c $\dfrac{1}{2\cos^2\theta} \equiv \dfrac{1}{2}\left(\dfrac{1}{\cos\theta}\right)^2 \equiv \dfrac{1}{2}\sec^2\theta$ ✓

d $\dfrac{1-\sin^2\theta}{\sin^2\theta} \equiv \dfrac{\cos^2\theta}{\sin^2\theta} = \left(\dfrac{\cos\theta}{\sin\theta}\right)^2 = \cot^2\theta$ ✓

e $$\frac{\sec\theta}{\cos^4\theta} \equiv \left(\frac{1}{\cos\theta}\right)\left(\frac{1}{\cos^4\theta}\right) \equiv \left(\frac{1}{\cos\theta}\right)^5 \equiv \sec^5\theta \checkmark$$

f $$\sqrt{\operatorname{cosec}^3\theta \cot\theta \sec\theta} \equiv \sqrt{\left(\frac{1}{\sin^3\theta}\right)\left(\frac{\cancel{\cos\theta}}{\sin\theta}\right)\left(\frac{1}{\cancel{\cos\theta}}\right)} \equiv \sqrt{\left(\frac{1}{\sin^4\theta}\right)}$$

$$\equiv \left(\frac{1}{\sin\theta}\right)^2 \equiv \operatorname{cosec}^2\theta \checkmark$$

g $$\frac{2}{\sqrt{\tan\theta}} \equiv 2\left(\frac{1}{\tan\theta}\right)^{1/2} \equiv 2\cot^{1/2}\theta \checkmark$$

h $$\frac{\operatorname{cosec}^2\theta \tan^2\theta}{\cos\theta} \equiv \left(\frac{1}{\cancel{\sin^2\theta}}\right)\left(\frac{\cancel{\sin^2\theta}}{\cos^2\theta}\right)\left(\frac{1}{\cos\theta}\right) \equiv \left(\frac{1}{\cos\theta}\right)^3 \equiv \sec^3\theta \checkmark$$

2 Write down values of $\cot x$ in each of these equations

a $5\sin x = 4\cos x$

$\Rightarrow \dfrac{5}{4} = \dfrac{\cos x}{\sin x}$

$\Rightarrow \cot x = 5/4$

b $\tan x = -2$

$\Rightarrow \dfrac{1}{\tan x} = \dfrac{1}{-2}$

$\Rightarrow \cot x = -1/2$

c $\quad 3\,\dfrac{\sin x}{\cos x} = \dfrac{\cos x}{\sin x}$

$\Rightarrow \quad 3 = \left(\dfrac{\cos x}{\sin x}\right)^2$

$\Rightarrow \quad \cot x = \pm\sqrt{3}$

3 Using the definitions of sec, cosec, cot and tan, simplify:

a $\quad \sin\theta \cot\theta \equiv \cancel{\sin\theta}\,\dfrac{\cos\theta}{\cancel{\sin\theta}} \equiv \cos\theta$

b $\quad \tan\theta \cot\theta \equiv \dfrac{\sin\theta}{\cos\theta} \times \dfrac{\cos\theta}{\sin\theta} \equiv 1$

c $\quad \tan 2\theta \operatorname{cosec} 2\theta \equiv \dfrac{\cancel{\sin 2\theta}}{\cos 2\theta} \times \dfrac{1}{\cancel{\sin 2\theta}} \equiv \sec 2\theta$

d $\cos\theta \sin\theta (\cot\theta + \tan\theta) \equiv \cos\theta \cancel{\sin\theta} \frac{\cos\theta}{\cancel{\sin\theta}} + \cancel{\cos\theta} \sin\theta \frac{\sin\theta}{\cancel{\cos\theta}}$

$$\equiv \cos^2\theta + \sin^2\theta$$

$$\equiv 1$$

e $\sin^3 x \operatorname{cosec} x + \cos^3 x \sec x \equiv \frac{\sin^3 x}{\sin x} + \frac{\cos^3 x}{\cos x}$

$$\equiv \sin^2 x + \cos^2 x$$

$$\equiv 1$$

f $\sec A - \sec A \sin^2 A \equiv \frac{1}{\cos A} - \frac{1}{\cos A} \sin^2 A$

$$\equiv \frac{1}{\cos A} (1 - \sin^2 A)$$

$$\equiv \frac{1}{\cos A} \cos^2 A = \cos A$$

g $\sec^2 x \cos^5 x + \cot x \operatorname{cosec} x \sin^4 x$

$$= \frac{1}{\cos^2 x}\cos^5 x + \frac{\cos x}{\sin x} \times \frac{1}{\sin x} \times \sin^4 x$$

$$\equiv \cos^3 x + \cos x \sin^2 x$$

$$\equiv \cos x \left(\cos^2 x + \sin^2 x\right)$$

$$\equiv \cos x \,(1)$$

$$\equiv \cos x$$

4 Show that

a $\cos\theta + \sin\theta\tan\theta \equiv \sec\theta$

$$\text{LHS} \equiv \cos\theta + \sin\theta\tan\theta$$

$$\equiv \frac{\cos^2\theta}{\cos\theta} + \sin\theta\,\frac{\sin\theta}{\cos\theta}$$

$$\equiv \frac{1}{\cos\theta}\left(\cos^2\theta + \sin^2\theta\right)$$

$$\equiv \sec\theta \equiv \text{RHS}. \quad \square$$

b Show that $\cot\theta + \tan\theta \equiv \operatorname{cosec}\theta \sec\theta$

$$\begin{aligned}
LHS &\equiv \cot\theta + \tan\theta \\
&\equiv \frac{\cos\theta}{\sin\theta} + \frac{\sin\theta}{\cos\theta} \\
&\equiv \frac{\cos^2\theta}{\sin\theta\cos\theta} + \frac{\sin^2\theta}{\sin\theta\cos\theta} \\
&\equiv \frac{\cos^2\theta + \sin^2\theta}{\sin\theta\cos\theta} \\
&\equiv \frac{1}{\sin\theta\cos\theta} \\
&\equiv \operatorname{cosec}\theta \sec\theta \equiv RHS \; \square.
\end{aligned}$$

c show that $\operatorname{cosec}\theta - \sin\theta \equiv \cos\theta\cot\theta$

$$\text{LHS} \equiv \operatorname{cosec}\theta - \sin\theta$$

$$\equiv \frac{1}{\sin\theta} - \sin\theta$$

$$\equiv \frac{1}{\sin\theta} - \frac{\sin^2\theta}{\sin\theta}$$

$$\equiv \frac{1-\sin^2\theta}{\sin\theta}$$

$$\equiv \frac{\cos^2\theta}{\sin\theta}$$

$$\equiv \cos\theta\left(\frac{\cos\theta}{\sin\theta}\right) \equiv \cos\theta\cot\theta \equiv \text{RHS}. \square.$$

4d $(1 - \cos x)(1 + \sec x) \equiv \sin x \tan x$

$$\begin{aligned}
\text{LHS} &\equiv (1 - \cos x)(1 + \sec x) \\
&\equiv 1 - \cos x + \sec x - \cos x \sec x \\
&\equiv \cancel{1} - \cos x + \sec x - \cancel{1} \\
&\equiv \sec x - \cos x \\
&\equiv \frac{1}{\cos x} - \frac{\cos^2 x}{\cos x} \\
&\equiv \frac{1 - \cos^2 x}{\cos x} \\
&\equiv \frac{\sin^2 x}{\cos x} \equiv \sin x \left(\frac{\sin x}{\cos x}\right) \equiv \sin x \tan x \equiv \text{RHS}. \ \square
\end{aligned}$$

4e show that $\dfrac{\cos x}{1-\sin x} + \dfrac{1-\sin x}{\cos x} \equiv 2\sec x$

$$\text{LHS} \equiv \frac{\cos x}{1-\sin x} + \frac{1-\sin x}{\cos x}$$

$$\equiv \frac{\cos^2 x}{(1-\sin x)\cos x} + \frac{(1-\sin x)^2}{(1-\sin x)\cos x}$$

$$\equiv \frac{\cos^2 x + 1 - 2\sin x + \sin^2 x}{(1-\sin x)\cos x}$$

$$\equiv \frac{2-2\sin x}{(1-\sin x)\cos x} \equiv \frac{2\cancel{(1-\sin x)}}{\cos x\cancel{(1-\sin x)}} \equiv 2\sec x \equiv \text{RHS}. \square$$

4f $$\frac{\cos\theta}{1+\cot\theta} \equiv \frac{\sin\theta}{1+\tan\theta}$$

$$\text{LHS} \equiv \frac{\cos\theta}{1+\cot\theta}$$

$$\equiv \frac{\cos\theta}{\frac{\sin\theta}{\sin\theta}+\frac{\cos\theta}{\sin\theta}}$$

$$\equiv \frac{\cos\theta\ \sin\theta}{\sin\theta+\cos\theta}$$

$$\equiv \frac{\cancel{\cos\theta}\ \sin\theta}{\cancel{\cos\theta}\left(\frac{\sin\theta}{\cos\theta}+1\right)} \equiv \frac{\sin\theta}{1+\tan\theta} \equiv \text{RHS}. \ \square$$

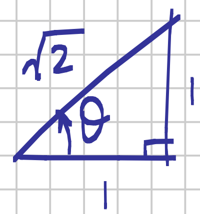
5 Solve for values of $\theta$ in the interval $0 \le \theta \le 360°$ the following equations (use 3 s.f.)

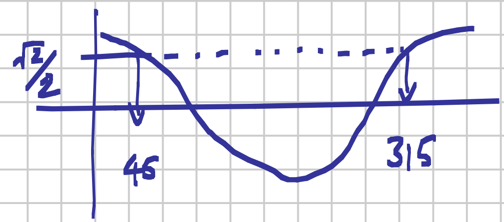
a $\sec\theta = \sqrt{2}$

$\Rightarrow \dfrac{1}{\cos\theta} = \sqrt{2}$

$\Rightarrow \cos\theta = \dfrac{1}{\sqrt{2}} = \dfrac{\sqrt{2}}{2}$

$\Rightarrow \cos\theta = 45°$ or $315°$



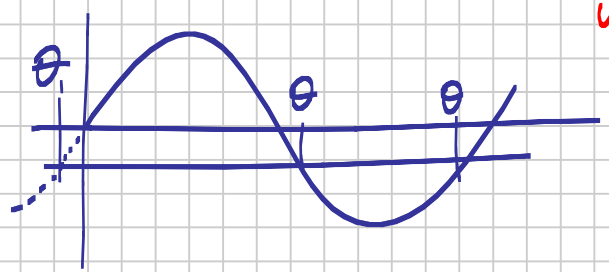
b $\operatorname{cosec}\theta = -3$

$$\frac{1}{\sin\theta} = -3$$

$$\sin\theta = \frac{-1}{3}$$

$$\theta_1 = \arcsin\left(-\frac{1}{3}\right)$$

in our chapter preview we agreed that from now on we'd replace $\sin^{-1}(x)$ with $\arcsin(x)$.




$$\theta_1 = -19.47122\ldots$$

$\therefore$ solutions at

$$\theta = 180 + 19.47122 \qquad \theta \simeq 199°$$

or

and $$\theta = 360 - 19.47122 \qquad \theta \simeq 341°$$

5c

$$5\cot\theta = -2$$

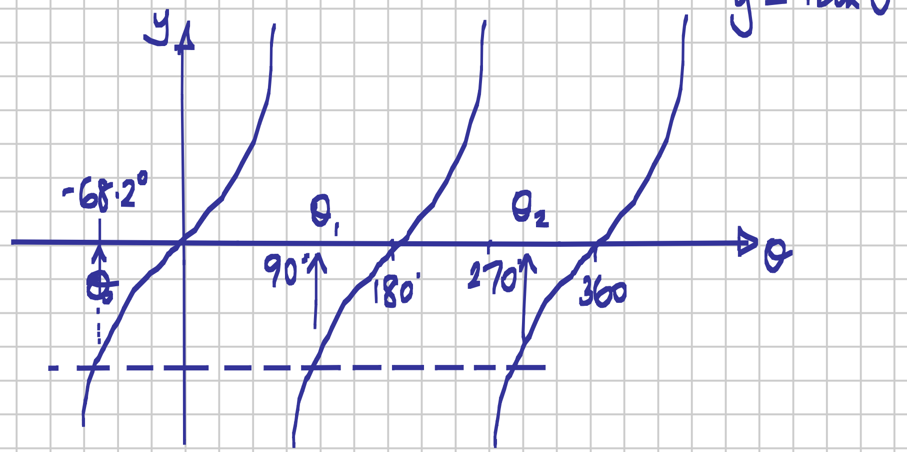
$$\frac{5}{\tan\theta} = -2$$

$$\Rightarrow \quad -\frac{5}{2} = \tan\theta$$

$$\theta_0 = -68.19859051$$

$$\theta_1 = 180 - 68.2^\circ \simeq 112^\circ$$

$$\theta_2 = 360 - 68.2^\circ \simeq 292^\circ$$



5d $\operatorname{cosec}\theta = 2$

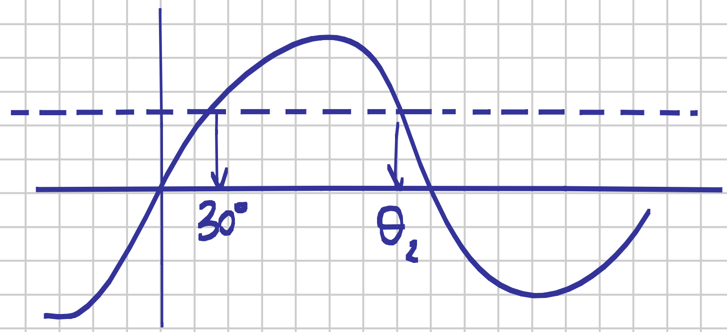
$$\Rightarrow \frac{1}{\sin\theta} = 2$$

$$\Rightarrow \sin\theta = \frac{1}{2}$$

$$\Rightarrow \theta_1 = \arcsin\left(\frac{1}{2}\right)$$

$$\theta_1 = 30°$$

$$\theta_2 = 180 - \theta_1 = 150°$$



5e $3\sec^2\theta - 4 = 0$

$\Rightarrow \sec^2\theta = \frac{4}{3}$

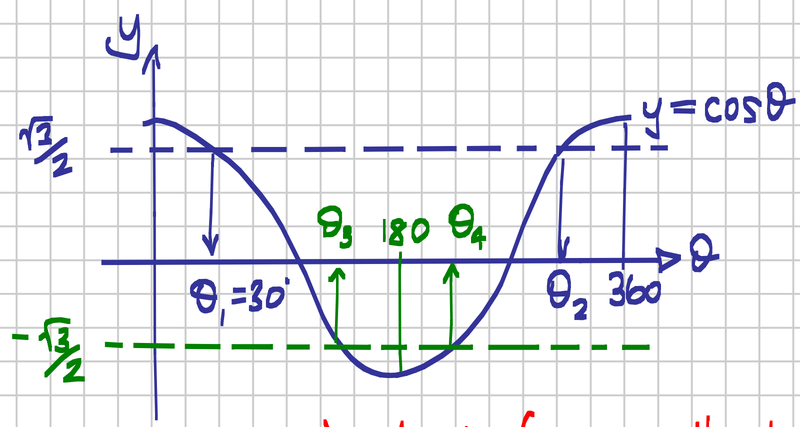
$\Rightarrow \frac{1}{\cos^2\theta} = \frac{4}{3}$

$\Rightarrow \cos^2\theta = \frac{3}{4}$

$\Rightarrow \cos\theta = \pm\frac{\sqrt{3}}{2}$

$\Rightarrow \theta_1 = 30°$

$\theta_2 = 360 - \theta_1 = 330°$




⚠ don't forget that $\sqrt{\frac{3}{4}}$ can be either positive or negative

$\theta_3 = \arccos\left(-\frac{\sqrt{3}}{2}\right) = 180 - \theta_1$ ($\cos^{-1}$)

$\theta_3 = 150°$

$\theta_4 = 180 + \theta_1 = 210°$

so there are four solutions $30°$, $150°$, $210°$ and $330°$.

5f $5\cos\theta = 3\cot\theta$

$\Rightarrow 5\cos\theta = \dfrac{3\cos\theta}{\sin\theta}$

$\Rightarrow 5\cos\theta\sin\theta = 3\cos\theta$

$\Rightarrow 5\cos\theta\sin\theta - 3\cos\theta = 0$

$\Rightarrow \cos\theta(5\sin\theta - 3) = 0$

This is the lightbulb idea: bring it all to one side & factorise. You MUSTN'T just cancel the $\cos\theta$: if you do, you'll lose solutions.

So either $\cos\theta = 0$

or $5\sin\theta - 3 = 0$

When $\cos\theta = 0 \Rightarrow \theta = 90°$ or $270°$

$5\sin\theta - 3 = 0 \Rightarrow \sin\theta = \frac{3}{5} \Rightarrow \theta_1 = 36.86989765... \approx 36.9°$

$\theta_2 = 180 - \theta_1 = 143°$

four solutions: $\theta = 36.9°, 90°, 143°, 270°$.

5g $\cot^2\theta - 8\tan\theta = 0$

$\Rightarrow \dfrac{1}{\tan^2\theta} = 8\tan\theta$

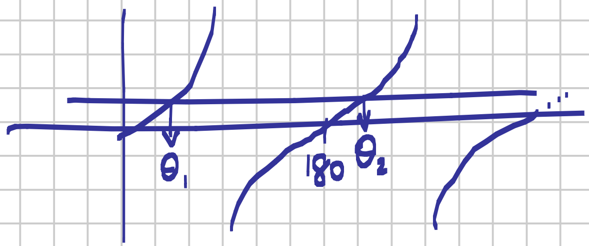
$\Rightarrow \dfrac{1}{8} = \tan^3\theta$

$\Rightarrow \tan\theta = \sqrt{\dfrac{1}{8}} = \dfrac{1}{2}$

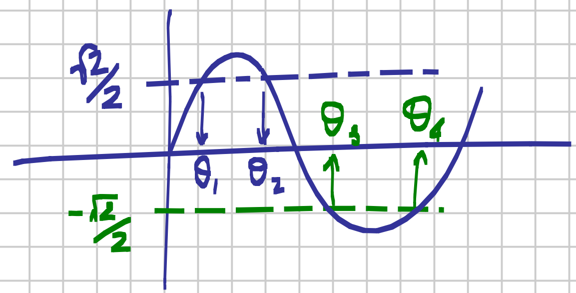
$\Rightarrow \theta = \arctan\left(\dfrac{1}{2}\right) \xleftarrow{\tan^{-1}}$

$\theta_1 = 26.56505118\ldots \simeq 26.6^\circ$

$\theta_2 \simeq 207^\circ$



5h $2\sin\theta = \operatorname{cosec}\theta$




$\Rightarrow 2\sin\theta = \frac{1}{\sin\theta}$

$\Rightarrow \sin^2\theta = \frac{1}{2}$

$\Rightarrow \sin\theta = \pm\sqrt{\frac{1}{2}} = \pm\frac{1}{\sqrt{2}}$ or $\pm\frac{\sqrt{2}}{2}$

$\theta_1 = \arcsin\left(\frac{\sqrt{2}}{2}\right) = 45°$

$\theta_2 = 180 - \theta_1 = 135°$

$\theta_3 = 180 + \theta_1 = 225°$

$\theta_4 = 360 - \theta_1 = 315°$

So the four solutions are 45°, 135°, 225°, 315°.

6 Solve for $-180° \le \theta \le 180°$ ...

a $\operatorname{cosec}\theta = 1$

$\Rightarrow \dfrac{1}{\sin\theta} = 1$


$\Rightarrow \sin\theta = 1$

$\Rightarrow \theta = 90°$ (only solution!)

6b

$$\sec\theta = -3$$

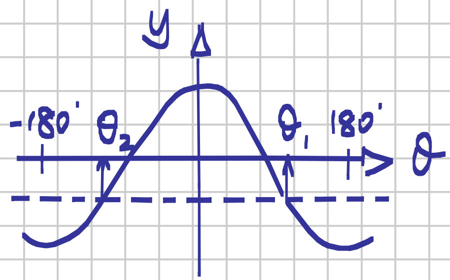
$$\Rightarrow \frac{1}{\cos\theta} = -3$$

$$\Rightarrow \cos\theta = \frac{-1}{3}$$

$$\Rightarrow \theta_1 = \arccos\left(\frac{-1}{3}\right)$$

$$\theta_1 = 109.4712206...^\circ \approx 109^\circ$$

$$\theta_2 = -\theta_1 = -109^\circ$$

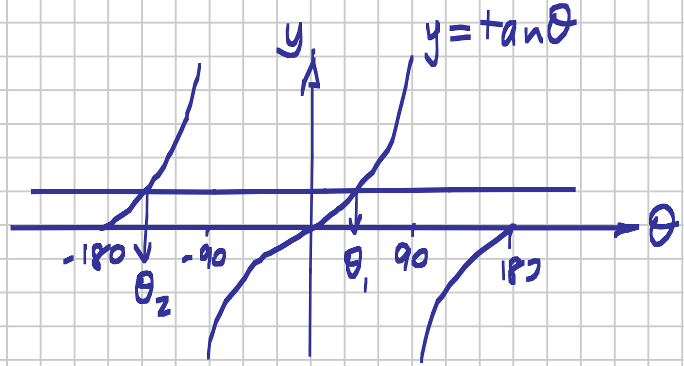


6c

$$\cot\theta = 3.45$$

$$\frac{1}{\tan\theta} = \frac{345}{100}$$

$$\Rightarrow \tan\theta = \frac{100}{345} = \frac{20}{69}$$




$$\theta_1 = \arctan\left(\frac{20}{69}\right)$$

$$\theta_1 = 16.16449915^\circ \simeq 16.2^\circ$$

$$\theta_2 = \theta_1 - 180 \simeq -164^\circ$$

6d $2\operatorname{cosec}^2\theta - 3\operatorname{cosec}\theta = 0$

$\Rightarrow \operatorname{cosec}\theta\,(2\operatorname{cosec}\theta - 3) = 0$

factorise and put each factor $= 0$.

$\Rightarrow$ either $\operatorname{cosec}\theta = 0$

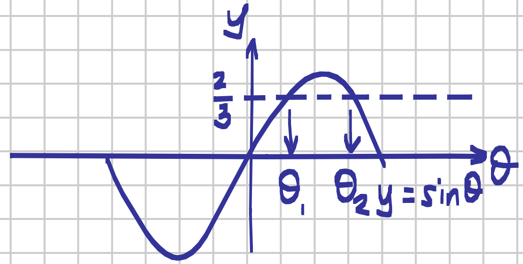
or $2\operatorname{cosec}\theta - 3 = 0$

Now there is no value of $\theta$ for which $\operatorname{cosec}\theta = 0$

So $2\operatorname{cosec}\theta - 3 = 0$

$\Rightarrow \operatorname{cosec}\theta = \frac{3}{2}$

$\Rightarrow \sin\theta = \frac{2}{3}$




$\Rightarrow \theta_1 = \arcsin\left(\frac{2}{3}\right) = 41.8103149^\circ \approx 41.8^\circ$

$\theta_2 = 180 - \theta_1 \approx 138^\circ$

6e

$$\sec\theta = 2\cos\theta$$

$$\Rightarrow \frac{1}{\cos\theta} = 2\cos\theta$$

$$\Rightarrow \frac{1}{2} = \cos^2\theta$$

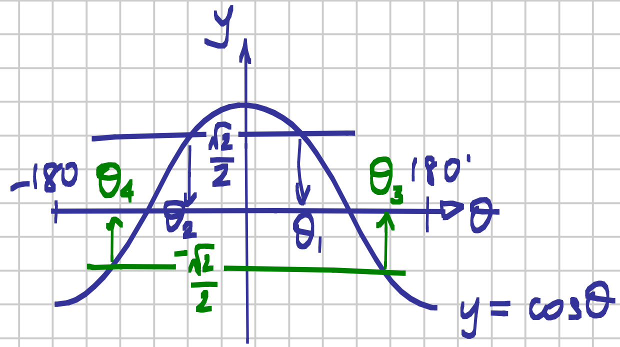
$$\Rightarrow \cos\theta = \pm\sqrt{1/2} = \pm\frac{\sqrt{2}}{2}$$

$$\theta_1 = \arccos\left(\frac{\sqrt{2}}{2}\right) = 45^\circ$$

$$\theta_2 = -45^\circ$$

$$\theta_3 = 90 + \theta_1 = 135^\circ$$

$$\theta_4 = -135^\circ$$



6f $3\cot\theta = 2\sin\theta$

$\Rightarrow \dfrac{3\cos\theta}{\sin\theta} = 2\sin\theta$

$\Rightarrow 3\cos\theta = 2\sin^2\theta$

I don't want a mixture of sin and cos if I can help it. How can I swap $\sin^2\theta$ for something else?

Now $\sin^2\theta + \cos^2\theta \equiv 1 \Rightarrow \sin^2\theta \equiv 1 - \cos^2\theta$

$\Rightarrow 3\cos\theta = 2(1-\cos^2\theta)$

$\Rightarrow 2\cos^2\theta + 3\cos\theta - 2 = 0$

$\Rightarrow (2\cos\theta - 1)(\cos\theta + 2) = 0$

$\Rightarrow 2\cos\theta - 1 = 0$ or $\cos\theta + 2 = 0$

$\Rightarrow \cos\theta = \frac{1}{2} \Rightarrow \theta = \pm 60^\circ$ (only solutions since $\cos\theta$ can't be $-2$)

It can be helpful at this stage to replace

$2\cos^2\theta + 3\cos\theta - 2 = 0$

with

$2c^2 + 3c - 2 = 0$

$\Rightarrow (2c-1)(c+2) = 0$

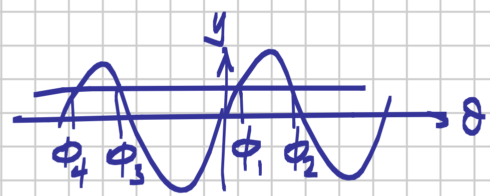
6g $\operatorname{cosec} 2\theta = 4$ in $-180 \leq \theta \leq 180$

$\Rightarrow \dfrac{1}{\sin 2\theta} = 4$ so $-360 \leq 2\theta \leq 360$

put $\phi = 2\theta$ then $-360° \leq \phi \leq 360°$.

$\Rightarrow \sin 2\theta = \dfrac{1}{4}$

$\Rightarrow \sin\phi = \dfrac{1}{4}$ and hence $\phi_1 = \arcsin\left(\frac{1}{4}\right)$




So $\phi_1 = 14.47751219 \simeq 14.5°$

$\phi_2 = 180 - \phi_1 \simeq 165.5°$

$\phi_3 = \phi_2 - 360 = -194.5°$

$\phi_4 = \phi_1 - 360 = -345.5°$

But since $\theta = \frac{1}{2}\phi$ we get $\theta_1 = 7.2°$, $\theta_2 = 82.8°$, $\theta_3 = -97.2°$ and $\theta_4 = -173°$

6h

$$2\cot^2\theta - \cot\theta - 5 = 0$$

Let $c = \cot\theta$

$$2c^2 - c - 5 = 0$$

This doesn't factorise (try though)

So use another method to solve the quadratic. I'm using completing the square.

$$2\left(c^2 - \tfrac{1}{2}c - \tfrac{5}{2}\right) = 0$$

$$\Rightarrow 2\left\{\left(c - \tfrac{1}{4}\right)^2 - \tfrac{1}{16} - \tfrac{40}{16}\right\} = 0$$

$$\Rightarrow \left(c - \tfrac{1}{4}\right)^2 = \tfrac{41}{16}$$

$$\Rightarrow c = \tfrac{1}{4} \pm \tfrac{\sqrt{41}}{4}$$

Now return to $\cot\theta$ : either $\cot\theta = \frac{1+\sqrt{41}}{4}$ or $\cot\theta = \frac{1-\sqrt{41}}{4}$

...